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LETTER TO THE EDITOR

A supersymmetric Frenet–Serret equation and supersymmetric nonlinear equations

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Abstract. A Frenet–Serret like equation is discussed in superspace. It is shown that new nonlinear supersymmetric equations can be generated which are completely integrable. As a particular case, the super sine-Gordon equation also belongs to this class.

In recent times a geometrical approach to nonlinear, completely integrable systems has become a very useful tool for discovering new equations and their Lax systems (Lamb 1977).

The Frenet–Serret equations of curves in space, when coupled with a suitable time evolution of the triad $(\boldsymbol{t}, \bar{N}, \bar{b})$ (in three-dimensional space; there are more vectors in higher-dimensional space), yield many nonlinear equations. Simultaneously, the study of supersymmetry in particle physics has given rise to many super nonlinear equations (D'Auria and Regge 1980, 1981), also completely integrable. Here we show how one can associate a moving frame of reference with such equations, and these equations look like a super Serret–Frenet equation. Some new equations are generated and the super-sine-Gordon belongs to this system as a particular case. Our rotating frame yields the Lax system of Girardello and Scuito (1978).

Let us denote by $\phi(x, \theta)$ a general superfield defined over ordinary space–time and anticommuting Grassmann coordinates θ. We assume for the present ϕ to be a scalar but having three components in the internal symmetry space, ϕ_1, ϕ_2, ϕ_3, collectively denoted by $\boldsymbol{\phi}(x, \theta)$. Let us choose two other vectors $\boldsymbol{e}_1(x, \theta)$ and $\boldsymbol{e}_2(x, \theta)$ such that

$$\boldsymbol{e}_i(x, \theta) = \varepsilon_{ij}(\boldsymbol{\phi}(x, \theta) \times \boldsymbol{e}_j(x, \theta)), \qquad \boldsymbol{e}_i \boldsymbol{e}_j = \delta_{ij}; \tag{1}$$

as x and θ change, this frame 'rotates', which is expressed through the following equations:

$$D_\alpha \boldsymbol{\phi} = -V^i_\alpha \boldsymbol{e}_i, \qquad D_\alpha \boldsymbol{e}_i = V^i_\alpha \boldsymbol{\phi} + \varepsilon^{ij} A_\alpha \boldsymbol{e}_j, \tag{2}$$

where

$$D_1 = -\partial/\partial\theta_2 + \mathrm{i}\theta_2 \partial/\partial x, \qquad D_2 = \partial/\partial\theta_1 + \mathrm{i}\theta_1 \partial/\partial t,$$

where V^i_α and A^i_α are supersymmetric coefficients to be chosen suitably. Now in the usual terminology of differential geometry, if such a moving frame is identified with the tangent, normal and the binormal system, then it is called a Frenet–Serret equation. Our system of equations closely resembles such a system and we may call it a

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'supersymmetric Frenet–Serret' system. However, an interesting difference persists. In the usual differential geometry one only has one set of equations like (2), which expresses the variations of the vectors with respect to arc length. To obtain nonlinear equations one usually invokes the time (Lamb 1977, Chinia 1980), which by proper choice of coefficients reduces to the known equations.

We now demand consistency of the equations (2), that is

$$D_2D_1(\boldsymbol{\phi}, \boldsymbol{e}_1, \boldsymbol{e}_2) = -D_1D_2(\boldsymbol{\phi}, \boldsymbol{e}_1, \boldsymbol{e}_2) \tag{3}$$

for each of the entries in the first bracket, which yields

$$\begin{aligned} &D_2v_1^1 + D_1v_2^1 = -[v_1^2A_2 + v_2^2A_1], \qquad D_2v_1^2 + D_1v_2^2 = v_2^1A_1 + v_1^1A_2, \\ &D_2A_1 + D_1A_2 = -v_2^1v_1^2 - v_1^1v_2^2. \end{aligned} \tag{4}$$

All other equations are identities. These are the coupled nonlinear supersymmetric equations.

Let us make the following choices for the coefficient functions.

$$v_1^1 = D_1u, \qquad A_2 = \cos u, \qquad v_2^2 = \sin u, \qquad A_1 = 0, \qquad v_1^2 - \beta \text{ (constant)};$$

then the system (4) becomes

$$D_2D_1u = -\beta \cos u \tag{5}$$

which is nothing but a variant of the super sine-Gordon equation.

Also let us set

$$\text{(i)} \qquad \left.\begin{aligned} v_2^1 = v_2^2 = \beta \\ A_2 = A_1\sigma \end{aligned}\right\} \qquad \beta \text{ constant;}$$

then we get

$$\begin{aligned} &D_2v_1^1 = -v_1^2A_1\sigma - \beta A_1, \qquad D_2v_1^2 = \beta A_1 + \sigma v_1^1 A_1, \\ &(D_2 + \sigma D_1)A_1 = -\beta(v_1^2 + v_1^1), \end{aligned} \tag{6}$$

which is something like a 'super three-wave equation'. Again, by choosing $v_2^1 = v_1^2 = \beta$ we obtain

$$\begin{aligned} &D_2v_1^1 = -\beta\sigma A_1 - v_2^2A_1, \qquad D_1v_2^2 = \beta A_1 + v_1^1\sigma A_1, \\ &(D_2 + \sigma D_1)A_1 = -\beta^2 - v_1^1v_2^2, \end{aligned} \tag{7}$$

which also belongs to the same class.

Besides the above facts, we note that our equations (2) do not contain any eigenvalue parameter λ. One may introduce such a parameter by invoking the invariance of the particular nonlinear equation and imposing that invariance on the 'IST' equations (2).

By writing them out in matrix form one may observe that our equations (2) in the case of the sine-Gordon system closely resemble those of Girardello and Scuito (1978).

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